

# Optical Turbulent Structures

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## Abstract

The problem of describing optical turbulent structures, arising in resonant media with high Fresnel numbers, is reviewed. The consideration is based on the probabilistic approach to pattern selection, ascribing a probability distribution of patterns to systems with multiple possible structures. The most probable structure corresponds to the minimal expansion exponent, or to the minimal expansion rate, characterizing the phase-space expansion of a dynamical system. Turbulent photon filamentation is studied. The most probable filament radius and the number of filaments are found, being in good agreement with experiment.

**Keywords:** spatio-temporal optical structures, optical turbulence, pattern selection, turbulent photon filamentation.

## 1. Experiments on Optical Structures

Spatio-temporal optical structures appear when electromagnetic waves propagate through a medium possessing some sort of nonlinearity. One can distinguish three types of such media, depending on the physical origin of nonlinearity.

(i) *Kerr medium*. This is a passive medium with a nonlinear relation between polarization and electric field, when susceptibility depends on the field intensity. Many liquid crystals pertain to this kind of matter. Different patterns arising in the Kerr medium are reviewed in Refs. [1–3].

(ii) *Photorefractive medium*. This pertains to a medium with a strong photorefractive effect. The photorefractive effect is the change in refractive index of a medium resulting from the optically induced redistribution of electrons and holes. The modulation of the medium refractive index is usually realized as a four-wave mixing process. Two input optical fields produce intensity grating, shifting electrons from donor atoms, which induces a space charge wave, equivalent to the formation of a ripple field. The nonlinear interaction of the input fields

with the medium leads to the generation of two output waves. An example of a photorefractive medium is the bismuth silicon oxide crystal  $\text{Bi}_{12}\text{SiO}_{20}$ . Various patterns are reviewed in Ref. [2,3].

(iii) *Resonant medium*. This, evidently, is an active medium typical of laser systems. Nonlinearity arises because of an effective interaction of resonant atoms through the common radiation field. Different spatio-temporal structures were observed in a  $\text{Na}_2$  laser,  $\text{CO}_2$  laser, dye laser and in many semiconductor and vapour lasers, as is reviewed in Refs. [3–6].

Pattern formation in nonlinear optics, which is also called optical morphogenesis, has many similarities for different media. This especially concerns the case when nonlinear optical samples have the same geometry. The most often used is the cylindrical geometry typical of lasers. Characteristic structures of cylindrical samples are filaments, aligned along the cylindrical axis, where the radiation intensity is much higher than in the space between these filaments. The appearance of such structures is easily noticeable in the transverse cross-section, where the high-intensity filaments are exhibited as bright spots. For brevity, we call these filamentary structures *photon filaments* [5,6].

In the case of cylindrical samples, the peculiar properties of photon filaments essentially depend on the Fresnel number

$$F \equiv \frac{\pi R^2}{\lambda L},$$

in which  $R$  is the cylinder radius,  $L$  is the characteristic length, and  $\lambda$  is the radiation wavelength. The Fresnel number is a sort of an aspect ratio comparing the transverse and longitudinal characteristics.

The Fresnel number in optics plays the same role as the Reynolds number in hydrodynamics. When increasing the Reynolds number, the fluid passes from laminar motion to turbulent one. A similar transition happens in optics when increasing the Fresnel number, and occurs around  $F \approx 10$ .

At small Fresnel numbers  $F < 5$ , there can exist several transverse modes with a regular behaviour. These transverse structures are regular in space, forming ordered geometric arrays seen as polygons in the transverse cross-section, with the number of bright spots proportional to  $F^2$ . The structures are also regular in time being either stationary or periodically oscillating. The type of such regular structures is prescribed by the sample geometry and corresponds to the empty-cavity Gauss-Laguerre modes imposed by the cylindrical geometry. This regular behaviour in optics is analogous to the laminar motion in hydrodynamics, and it is well understood theoretically both for nonlinear media, such as a Kerr medium and photorefractive crystals [1–3], and for laser media [4].

For large Fresnel numbers  $F > 15$ , the appearing spatio-temporal patterns are principally different from the empty-cavity modes. The modal expansion is no longer valid and the boundary conditions have no importance. The medium looks as a bunch of independently flashing filaments, whose number is proportional to  $F$ . The filaments are chaotically distributed in the transverse cross-section, are not correlated with each other, and are flashing aperiodically in time. This spatio-temporal optical chaos is similar to hydrodynamic turbulence, because of

which it is called *optical turbulence*. Since this phenomenon is characterized by the formation of bright filaments with a high photon density, it has been named [7] *turbulent photon filamentation*. This kind of optical turbulence was observed in nonlinear media, such as photorefractive crystals [2,3] as well as in different lasers, such as CO<sub>2</sub>, dye, and vapour lasers (see discussion in Refs. [5–7]).

The theory of turbulent photon filamentation was advanced in Ref. [7]. This theory is based on the probabilistic approach to pattern selection [6].

## 2. Probabilistic Pattern Selection

The problem of pattern selection arises when evolution equations possess several solutions corresponding to different spatio-temporal structures. This problem is rather common for many nonlinear systems, optical media being only a particular case [8]. Therefore, let us, first, formulate a general approach to pattern selection, relevant for any dynamical system.

Let us consider a set of functions, in general complex,  $y_k(x, t)$  enumerated by the index  $k = 1, 2, \dots$  and depending on a collection of variables  $x \in \mathbb{D} \subset \mathbb{R}^d$  and on time  $t \in \mathbb{R}$ . The set of  $y(t) \equiv \{y_k(x, t)\}$  is the *dynamical state*. For the compactness of notation, it is convenient to denote by one index  $i$  the pair  $k$  and  $x$ , writing  $y(t) = \{y_i(t)\}$ , with  $y_i(t) \equiv y_k(x, t)$ . Then the dynamical state  $y(t)$  can be treated as a column with respect to  $i$ . In this notation, the system of evolution equations can be presented as

$$\frac{d}{dt} y(t) = v(y, t) , \quad (1)$$

with a velocity field being also a column  $v(y, t) = \{v_i(y, t)\}$ . Evolution equations containing only the first derivatives in time are called to have the *normal form*. Any system of equations containing higher time derivatives can always be reduced to the normal form [9].

The dynamical state  $y(t) \in \mathcal{F}$  pertains to a *phase space*  $\mathcal{F} = \otimes_i \mathcal{F}_i$ , being a tensor product of the spaces  $\mathcal{F}_i \supset y_i(t)$ . An *elementary phase volume* of the phase space  $\mathcal{F}$  can be written [6,7] as  $|\delta\Gamma(t)|$ , with

$$\delta\Gamma(t) \equiv \prod_i \delta y_i(t) . \quad (2)$$

The temporal behaviour of the phase volume element (2) is characterized by the *expansion exponent*

$$\sigma(t) \equiv \ln \left| \frac{\delta\Gamma(t)}{\delta\Gamma(0)} \right| , \quad (3)$$

which shows whether the phase volume expands with time or contracts according to the law

$$|\delta\Gamma(t)| = |\delta\Gamma(0)| e^{\sigma(t)} .$$

The expansion exponent (3) can be presented in several equivalent forms [6,7]. Introducing the *multiplier matrix*  $\hat{M}(t) = [M_{ij}(t)]$  with the elements

$$M_{ij}(t) \equiv \frac{\delta y_i(t)}{\delta y_j(0)} , \quad (4)$$

we can transform Eq. (3) to

$$\sigma(t) = \ln |\det \hat{M}(t)| . \quad (5)$$

Another form of the expansion exponent (3) can be obtained by means of the *expansion matrix*  $\hat{X}(t) = [X_{ij}(t)]$  with the elements

$$X_{ij}(t) \equiv \frac{\delta v_i(y, t)}{\delta y_j(t)} . \quad (6)$$

The variation of the evolution equation (1) yields

$$\frac{d}{dt} \hat{M}(t) = \hat{X}(t) \hat{M}(t) . \quad (7)$$

From Eqs. (6) and (7), we find

$$\sigma(t) = \text{Re} \int_0^t \text{Tr} \hat{X}(t') dt' . \quad (8)$$

If the multiplier matrix (4) possesses eigenvalues  $\mu_n(t)$  enumerated by a multi-index  $n$ , then Eq. (5) gives

$$\sigma(t) = \sum_n \ln |\mu_n(t)| . \quad (9)$$

Let us introduce the *local expansion rate*

$$\Lambda(t) \equiv \frac{1}{t} \sigma(t) . \quad (10)$$

The latter, depending on the form of the expansion exponent, can be written in the following three ways:

$$\Lambda(t) = \frac{1}{t} \ln |\det \hat{M}(t)| , \quad (11)$$

$$\Lambda(t) = \frac{1}{t} \text{Re} \int_0^t \text{Tr} \hat{X}(t') dt' , \quad (12)$$

$$\Lambda(t) = \sum_n \frac{1}{t} \ln |\mu_n(t)| . \quad (13)$$

From Eq. (13), we have

$$\lim_{t \rightarrow \infty} \Lambda(t) = \sum_n \lambda_n , \quad (14)$$

where the right-hand side is the sum of the Lyapunov exponents

$$\lambda_n \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \ln |\mu_n(t)| , \quad (15)$$

provided the corresponding limit exists.

It is easy to check that

$$\lim_{t \rightarrow \infty} \frac{d}{dt} \sigma(t) = \sum_n \lambda_n , \quad (16)$$

which immediately follows from Eqs. (9) and (15). Since from Eq. (8) one has

$$\frac{d}{dt} \sigma(t) = \text{Re Tr} \hat{X}(t) ,$$

then, according to Eq. (16), one gets

$$\lim_{t \rightarrow \infty} \text{Re Tr} \hat{X}(t) = \sum_n \lambda_n .$$

These limiting properties remind us those of the Gibbs entropy

$$S_G(t) = - \int \rho(y, t) \ln \rho(y, t) dy ,$$

in which  $\rho(y, t)$  is a probability density. For the Gibbs entropy in steady states, one has [10–14] the limit

$$\lim_{t \rightarrow \infty} \frac{d}{dt} S_G(t) = \sum_n \lambda_n .$$

The quantity  $dS_G/dt$  is called the entropy production rate or, simply, the entropy production. Therefore, the expansion exponent  $\sigma(t)$  is a kind of *dynamic entropy variation*, and the expansion rate  $\Lambda(t)$  resembles the entropy production rate. More details on the entropy production can be found in Refs. [11–16].

Notice that normally an invariant probability measure of a steady state is singular [13] and therefore the Gibbs entropy  $S_G(t)$  is  $S_G(t) = -\infty$ , which is physically unacceptable.

It is worth mentioning that in dynamical theory the quantity  $\text{Re Tr} \hat{X}(t)$  is sometimes termed the contraction rate. According to Eq. (12), the expansion rate (10) could also be named the contraction rate. The usage of the words "expansion" or "contraction" is rather philological. But, probably, it is more logical to call  $\Lambda(t)$  the expansion rate, since  $\Lambda(t) > 0$  means really the expansion of the phase volume  $|\delta\Gamma(t)|$ , while  $\Lambda(t) < 0$  implies the contraction of the latter.

Now let us turn to the problem of pattern selection, when the evolution equation (1) possesses a set of solutions corresponding to different spatio-temporal structures. If we label these different patterns by an index  $\alpha$ , then for each pattern we have a solution  $y(\alpha, t)$  and, respectively, an expansion exponent  $\sigma(\alpha, t)$  and an expansion rate  $\Lambda(\alpha, t)$ . The problem is how to classify these different solutions and the related patterns.

The probabilistic approach to pattern selection [6,7] is based on the idea that each pattern can be characterized by its probabilistic weight. That is, there should exist a pattern distribution  $p(\alpha, t)$ . The latter can be derived by minimizing the *pattern information*

$$I_p(t) \equiv \int p(\alpha, t) \ln p(\alpha, t) d\alpha + \int p(\alpha, t) \sigma(\alpha, t) d\alpha , \quad (17)$$

consisting of the sum of the Shanon information and lost information, under the normalization condition

$$\int p(\alpha, t) d\alpha = 1 .$$

This implies the minimization of the conditional pattern information

$$\tilde{I}_p(t) \equiv I_p(t) + l(t) \left[ \int p(\alpha, t) d\alpha - 1 \right] ,$$

in which  $l(t) \equiv \ln Z(t) - 1$  is a Lagrange multiplier. Minimizing  $\tilde{I}_p(t)$  results in the *pattern distribution*

$$p(\alpha, t) = \frac{1}{Z(t)} \exp\{-\sigma(\alpha, t)\} , \quad (18)$$

with the normalization factor

$$Z(t) = \int \exp\{-\sigma(\alpha, t)\} d\alpha .$$

Taking into account the relation (10), we obtain

$$p(\alpha, t) = \frac{1}{Z(t)} \exp\{-\Lambda(\alpha, t) t\} . \quad (19)$$

In this way, the *most probable pattern* corresponds to the *minimal expansion exponent*  $\sigma(\alpha, t)$  or, equivalently, to the *minimal expansion rate*  $\Lambda(\alpha, t)$ , that is, one has the relation

$$\min_{\alpha} \Lambda(\alpha, t) \Longleftrightarrow \max_{\alpha} p(\alpha, t) . \quad (20)$$

Thus, we come to the following conclusion.

**Principle of Pattern Selection.** The most probable pattern for a dynamical system at a given time is defined by the minimal expansion rate.

Let us emphasize that the formulated here approach of *Probabilistic Pattern Selection* is a very different from the Prigogine [17–21] principle of minimal entropy production in stationary states. The latter principle deals with the entropy production  $\dot{S} \equiv dS/dt$  defined as time derivative of thermodynamical entropy. It says that the entropy production  $\dot{S}(\alpha)$  in a stationary state is minimal with respect to internal thermodynamic variables  $\alpha$ . The basic points, distinguishing the *Probabilistic Pattern Selection* from the Prigogine principle, are as follows:

(i) This approach is developed for arbitrary dynamical systems. In particular, it may concern the evolution equations for thermodynamic variables or for statistical averages of some operators.

(ii) The theory is valid for any dynamical states, not necessarily stationary ones.

(iii) Not solely the most probable pattern is defined, but the probability distribution for all feasible patterns is suggested.

### 3. Turbulent Photon Filamentation

Now we shall apply the theory of Section 2 to the problem of turbulent photon filamentation, discussed in Section 1. Consider a system of  $N$  resonant atoms interacting with electromagnetic field as is described by the Hamiltonian

$$\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_{af} \quad (21)$$

consisting of the terms: The Hamiltonian of two-level resonant atoms

$$\hat{H}_a = \sum_{i=1}^N \omega_0 \left( \frac{1}{2} + S_i^z \right) , \quad (22)$$

where  $\omega_0$  is a transition frequency and  $S_i^z$  is a spin-1/2 operator. The radiation-field Hamiltonian

$$\hat{H}_f = \frac{1}{8\pi} \int \left( \mathbf{E}^2 + \mathbf{H}^2 \right) d\mathbf{r} , \quad (23)$$

with electric field  $\mathbf{E}$  and magnetic field  $\mathbf{H} = \nabla \times \mathbf{A}$ . The vector potential is assumed to satisfy the Coulomb gauge calibration  $\nabla \cdot \mathbf{A} = 0$ . The atom-field interaction Hamiltonian

$$\hat{H}_{af} = - \sum_{i=1}^N \left( \frac{1}{c} \mathbf{J}_i \cdot \mathbf{A}_i + \mathbf{D}_i \cdot \mathbf{E}_{0i} \right) , \quad (24)$$

in which  $\mathbf{A}_i \equiv \mathbf{A}(\mathbf{r}, t)$ , and the transition-current and transition-dipole operators are

$$\mathbf{J}_i = i\omega_0 \left( \mathbf{d} S_i^+ - \mathbf{d}^* S_i^- \right) , \quad \mathbf{D}_i = \mathbf{d} S_i^+ + \mathbf{d}^* S_i^- ,$$

where  $\mathbf{d}$  is the transition dipole,  $\mathbf{d} \equiv d_0 \mathbf{e}_d$ ;  $S_i^\pm$  are the rising or lowering operators, respectively; and  $\mathbf{E}_{0i}$  is a seed field.

We shall consider the evolution equations for the functions

$$u(\mathbf{r}, t) \equiv 2 \langle S^-(\mathbf{r}, t) \rangle , \quad s(\mathbf{r}, t) \equiv 2 \langle S^z(\mathbf{r}, t) \rangle , \quad (25)$$

being the statistical averages of the corresponding operators. Introduce the notation

$$f(\mathbf{r}, t) = f_0(\mathbf{r}, t) + f_{rad}(\mathbf{r}, t) \quad (26)$$

for an effective field acting on an atom, with the cavity seed field

$$f_0(\mathbf{r}, t) \equiv -2i\mathbf{d} \cdot \mathbf{E}_0(\mathbf{r}, t) \quad (27)$$

and the radiation field

$$f_{rad}(\mathbf{r}, t) \equiv -\frac{3}{4} i\gamma\rho \int \left[ \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{k_0|\mathbf{r}-\mathbf{r}'|} u(\mathbf{r}', t) - \mathbf{e}_d^2 \frac{e^{-ik_0|\mathbf{r}-\mathbf{r}'|}}{k_0|\mathbf{r}-\mathbf{r}'|} u^*(\mathbf{r}', t) \right] d\mathbf{r}' , \quad (28)$$

where  $k_0 \equiv \omega_0/c$ ,  $\rho$  is the density of atoms, and  $\gamma \equiv 4k_0^3 d_0^2/3$  is the natural width. The seed field

$$\mathbf{E}_0 = \frac{1}{2} \mathbf{E}_1 e^{i(kz-\omega t)} + \frac{1}{2} \mathbf{E}_1^* e^{-i(kz-\omega t)} \quad (29)$$

selects the longitudinal mode with the frequency  $\omega = kc$  satisfying the quasiresonance condition

$$\frac{|\Delta|}{\omega_0} \ll 1, \quad \Delta \equiv \omega - \omega_0.$$

The evolution equations for the atomic variables (25) can be derived [5] by eliminating the field variables and invoking the semiclassical approximation. This yields

$$\begin{aligned} \frac{\partial u}{\partial t} &= -(i\omega_0 + \gamma_2)u + fs, & \frac{\partial |u|^2}{\partial t} &= -2\gamma_2|u|^2 + (u^*f + f^*u)s, \\ \frac{\partial s}{\partial t} &= -\frac{1}{2}(u^*f + f^*u) - \gamma_1(s - \zeta), \end{aligned} \quad (30)$$

where  $\gamma_1$  is the level width,  $\gamma_2$  is the line width, and  $\zeta > 0$  is a pumping parameter.

Without the loss of generality, we may look for the solutions to Eqs. (30) in the form of a bunch of  $N_f$  filaments,

$$u(\mathbf{r}, t) = \sum_{n=1}^{N_f} u_n(r_\perp, t) e^{ikz}, \quad s(\mathbf{r}, t) = \sum_{n=1}^{N_f} s_n(r_\perp, t), \quad (31)$$

with  $r_\perp \equiv \sqrt{x^2 + y^2}$ . In the case of very small Fresnel numbers,  $F \ll 1$ , there is the sole filament, so that  $N_f = 1$  and  $u_n$  and  $s_n$  are uniform in space. For low Fresnel numbers  $F < 10$ , when the regime of laminar filamentation occurs, an expansion over Gauss-Laguerre modes is more appropriate [4]. And for large Fresnel number  $F > 10$ , when the regime of turbulent filamentation develops, there appear many uncorrelated filaments. Then the functions  $u_n$  and  $s_n$  are essentially nonzero only around the axis of an  $n$ -th filament, but fastly decrease outside the filament, so that  $u_m u_n \sim \delta_{mn}$ ,  $s_m s_n \sim \delta_{mn}$ , and  $u_m s_n \sim \delta_{mn}$ . Here we are interested in the latter case of turbulent photon filamentation.

To describe the turbulent regime, let us introduce the averaged functions

$$u_n(t) \equiv \frac{1}{V_n} \int_{V_n} u_n(r_\perp, t) d\mathbf{r}, \quad s_n(t) \equiv \frac{1}{V_n} \int_{V_n} s_n(r_\perp, t) d\mathbf{r}, \quad (32)$$

with the averaging over a cylinder enveloping the  $n$ -th filament. The volume of the enveloping cylinder is  $V_n = \pi a_n^2 L$ , with  $a_n$  being the cylinder radius and  $L$ , the sample length. The radius of the enveloping cylinder  $a_n$  is related to the filament radius  $r_n$  by the *conservation-energy equation*

$$\int |u_n(r_\perp, t)|^2 d\mathbf{r} = V_n |u_n(r_n, t)|^2. \quad (33)$$



If the filament profile is well approximated by the normal law  $\exp(-r_{\perp}^2/2r_n^2)$ , the relation (33) gives

$$a_n = 1.82 r_n . \quad (34)$$

The sample is assumed to have the cylindrical shape of radius  $R$  and length  $L$ . For these values and for the radiation wavelength  $\lambda$ , the inequalities

$$\lambda \ll R \ll L ,$$

typical of lasers, are valid. For the functions (32), characterizing the turbulent regime, we obtain the equations

$$\begin{aligned} \frac{du_n}{dt} &= -(i\Omega_n + \Gamma_n)u_n + f_1 s_n , & \frac{d|u_n|^2}{dt} &= -2\Gamma_n|u_n|^2 + (u_n^* f_1 + f_1^* u_n)s_n , \\ \frac{ds_n}{dt} &= -g_n \gamma_2 |u_n|^2 - \frac{1}{2} (u_n^* f_1 + f_1^* u_n) - \gamma_1 (s_n - \zeta) , \end{aligned} \quad (35)$$

where

$$\Gamma_n \equiv \gamma_2(1 - g_n s) , \quad \Omega_n \equiv \omega_0 + \gamma_2 g'_n , \quad f_1 \equiv -i\mathbf{d} \cdot \mathbf{E}_1 e^{-i\omega t} ,$$

and the effective *coupling parameters* are

$$g_n \equiv \frac{3\gamma\rho}{4\gamma_2} \int_{V_n} \frac{\sin(k_0 r - kz)}{k_0 r} d\mathbf{r} , \quad g'_n \equiv \frac{3\gamma\rho}{4\gamma_2} \int_{V_n} \frac{\cos(k_0 r - kz)}{k_0 r} d\mathbf{r} . \quad (36)$$

The equations for different filaments are naturally decoupled since the filaments are not correlated with each other.

The nonlinear system of equations (35) can be solved by means of the scale separation approach [5,22]. The details of this solution can be found in Refs. [6,7]. Calculating the expansion rate (12) and minimizing it with respect to the filament radius, we obtain the most probable radius

$$r_f = 0.3\sqrt{\lambda L} , \quad (37)$$

which is in good agreement with experiment, as is discussed in refs. [5–7]. The formula (37) has been specially checked in the regime of turbulent photon filamentation of CO<sub>2</sub> and dye lasers and found to well describe the experimental observations [23–27].

The most probable number of filaments can be estimated from the normalization integral

$$\frac{1}{V} \int s d\mathbf{r} = \zeta , \quad (38)$$

where the integration is over the whole volume of the sample  $V = \pi R^2 L$ . Considering the developed turbulent regime, when the population difference inside each filament of radius  $r_f$  has reached the value of the pumping parameter  $\zeta$  and the population difference outside the filaments is  $s_{out}$ , we have from the integral (38) the equation

$$N_f V_f \zeta + (V - N_f V_f) s_{out} = \zeta V .$$

From here, we find

$$N_f = \frac{V}{V_f} = \left( \frac{R}{r_f} \right)^2 . \quad (39)$$

With the filament radius (37), this gives

$$N_f \cong 4F , \quad \frac{r_f}{R} \simeq \frac{1}{2\sqrt{F}} , \quad (40)$$

which also is in good agreement with experiment.

Filamentation in laser media is a rather general process and should arise for different samples, provided the Fresnel number is sufficiently high. This should concern as well the so-called random lasers [28].

The theory presented above has to do with the well developed optical turbulence corresponding to large Fresnel numbers. An interesting question is how to describe the intermediate region, where  $5 \leq F \leq 15$ , and the arising turbulence is yet weak, being intermittent with the remnants of the regular filaments, characterized by the Gauss-Laguerre modes. Then between the laminar and turbulent filaments there could appear a kind of Josephson interference, as it happens for electromagnetic vortices [29,30]. In the frame of the probabilistic pattern selection, described in Sec. 2, we could treat laminar and turbulent filaments as independent, calculating their probabilities. In the intermediate region of Fresnel numbers, where these two types of filaments coexist, their probabilities should be comparable. At low Fresnel numbers, the probability of the laminar filament must prevail, while at high  $F$ , the probability of the turbulent filaments is to become much larger. In this way, the whole picture for arbitrary  $F$ , varying in the interval  $[0, \infty)$ , can be done in the frame of the probabilistic pattern selection.

## References

- [1] S.A. Akhmanov, R.V. Khokhlov, and A.P. Sukhorukov, in *Laser Handbook*, edited by F.T. Arecchi and E. Shultz-DuBois (North-Holland, Amsterdam, 1972), V. 2, p. 1151.
- [2] F.T. Arecchi, *Physica D* **86**, 297 (1995).
- [3] F.T. Arecchi, S. Boccaletti, and P.L. Ramazza, *Phys. Rep.* **318**, 1 (1999).
- [4] L.A. Lugiato, *Phys. Rep.* **219**, 293 (1992).
- [5] V.I. Yukalov and E.P. Yukalova, *Phys. Part. Nucl.* **31**, 561 (2000).
- [6] V.I. Yukalov, *Physica A* **291**, 255 (2001).
- [7] V.I. Yukalov, *Phys. Lett. A* **278**, 30 (2000).
- [8] M.C. Cross and P.C. Hohenberg, *Rev. Mod. Phys.* **65**, 851 (1993).

- [9] N.N. Bogolubov and B.I. Sadovnikov, Mosc. Univ. Phys. Bull. **3**, 24 (1961).
- [10] N.I. Chernov, G.I. Eyink, J.L. Lebowitz, and Y.G. Sinai, Commun. Math. Phys. **154**, 569 (1993).
- [11] D. Ruelle, J. Stat. Phys. **85**, 1 (1996).
- [12] W. Breymann, T. Tel, and J. Volmer, Phys. Rev. Lett. **77**, 2945 (1996).
- [13] D. Ruelle, J. Stat. Phys. **95**, 393 (1999).
- [14] L. Rondoni and E.G.D. Cohen, Nonlinearity **13**, 1905 (2000).
- [15] R. Luzzi and A.R. Vasconcellos, Fortschr. Phys. **38**, 887 (1990).
- [16] J.G. Ramos, A.R. Vasconcellos, and R. Luzzi, Fortschr. Phys. **43**, 265 (1995).
- [17] I. Prigogine, *Introduction to Nonequilibrium Thermodynamics* (Wiley, New York, 1962).
- [18] I. Prigogine, *Nonequilibrium Statistical Mechanics* (Wiley, New York, 1962).
- [19] P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability and Fluctuations* (Wiley, New York, 1971).
- [20] W. Ebeling, *Strukturbildung bei Irreversiblen Prozessen* (Teubner, Leipzig, 1976).
- [21] Y.L. Klimontovich, *Turbulent Motion and Structure of Chaos* (Nauka, Moscow, 1990).
- [22] V.I. Yukalov, Phys. Rev. B **53**, 9232 (1996).
- [23] I. Pastor and J.M. Guerra, Appl. Phys. B **51**, 342 (1990).
- [24] I. Pastor, V.M. Pérez-Garcia, F. Encinas-Sanz, J.M. Guerra, and L. Vazquez, Physica D **66**, 412 (1993).
- [25] V.M. Pérez-Garcia and J.M. Guerra, Phys. Rev. A **50**, 1646 (1994).
- [26] V.M. Pérez-Garcia, I. Pastor, and J.M. Guerra, Phys. Rev. A **52**, 2392 (1995).
- [27] F. Encinas-Sanz, J.M. Guerra, and I. Pastor, Opt. Lett. **21**, 1153 (1996).
- [28] X. Jiang and C.M. Soukoulis, Phys. Rev. Lett. **85**, 70 (2000).
- [29] A. Vourdas, Europhys. Lett. **48**, 201 (1999).
- [30] A. Vourdas, Phys. Rev. B **60**, 620 (1999).